GEOPHYSICAL RESEARCH LETTERS, VOL. 35, L23502, doi:10.1029/2008GL035788, 2008

Avalanche crown-depth distributions

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Received 23 August 2008; revised 25 October 2008; accepted 3 November 2008; published 10 December 2008.

[1] The literature disagrees about the statistical distribution of snow avalanche crown depths. Large datasets from Mammoth Mountain, California and the Westwide Avalanche Network show that the three-parameter generalized extreme value distribution provides the most robust fit, followed by a two-parameter variation, the Fréchet distribution. The most parsimonious explanation is neither self-organized criticality nor other complex cascades, but the maximum domain of attraction, implying that distributions of individual avalanche crown depths are scaling. We also show that crown depths do not have a universal tail index. Rather, they range from 2.8 to 4.6 over different avalanche paths, consistent with other geophysical phenomena such as wildfires, which show similar variability. **Citation:** Bair, E. H., J. Dozier, and K. W. Birkeland (2008), Avalanche crown-depth distributions, *Geophys. Res. Lett.*, *35*, L23502, doi:10.1029/2008GL035788.

1. Introduction

[2] Different authors have modeled avalanche size distributions using distributions belonging to the subexponential class. Subexponentials have tails that decay more slowly than the normal distribution and are used to model highly variable data. Questions about best fit and tail-indices have not been resolved. We build on previous work involving subexponential distributions and apply it to the ongoing debate in the snow science community over avalanche-size distributions. We examine two very large data sets of avalanche crown depths to determine which distributions provide the best fit.

[3] After discussing scaling distributions, we review previously published studies on size distributions in snow avalanches. We then describe the Mammoth Mountain and Westwide Avalanche Network datasets, before presenting and discussing our results and proposing a generating mechanism for the observed distribution.

1.1. Scaling Distributions

[4] A subexponential distribution [*Goldie and Klüppelberg*, 1998] of a random variable $\bar{F}(x)$ has a right tail that decays more slowly than an exponential:

$$\frac{\bar{F}(x)}{e^{-\mu x}} \rightarrow \infty \text{ for all } \mu > 0 \qquad (1)$$

[5] Members of the subexponential family include the Pareto of first and second kind, Burr, lognormal, Weibull, and log-gamma. Gaussian and exponential distributions are not in the subexponential class since their tails decay as exponentials. Recent research has focused on whether avalanche size distributions belong to a subclass of sub-exponentials called scaling distributions. A distribution is scaling [*Mandelbrot*, 1982] if for some tail-index $\alpha > 0$ its survivor function $P(\geq x)$ follows a power law:

$$P(\geq x) = \int_x^\infty \mathrm{pdf}(x)dx \propto x^{-\alpha}, 0 < x < \infty \qquad (2)$$

[6] A scaling distribution lacks inherent scale: it is self-similar at all scales, and its sole response to conditioning is a change of scale [*Willinger et al.*, 2004]. Scaling distributions are invariant under aggregation and weighted mixture, and they are the only distributions that are invariant under maximization [*Mandelbrot*, 1997]. These strong invariance properties have led to the idea of scaling distributions being "more normal than normal" [*Willinger et al.*, 2004; *Brookings et al.*, 2005] as the most parsimonious model for highly variable data just as Gaussians or exponentials are for data with low variability.

1.2. Scaling Distributions, SOC, and HOT

[7] Scaling distributions have been cited as evidence of emergent behavior such as Self-Organized Criticality (SOC) [*Bak et al.*, 1988], systems driven towards a self-sustaining critical state. An alternative explanation for power laws is Highly Optimized Tolerance (HOT) [*Carlson and Doyle*, 1999, 2000]. In this view, power laws arise by system design, whereby systems are robust to common perturbations but fragile to rare events.

2. Literature Review

[8] Power-law behavior has been observed in avalanches throughout the Western US and internationally, using different size measures including crown depth [*Rosenthal and Elder*, 2003; *Faillettaz et al.*, 2006], class size [*Birkeland and Landry*, 2002], and area [*Louchet et al.*, 2002]. The ranges of sizes over which fits have been applied have been inconsistent, and *McClung* [2003] asserts that avalanche crown-depth distributions are lognormal instead. Proposed distribution generating mechanisms include chaotic processes [*Rosenthal and Elder*, 2003], self-organized criticality [*Birkeland and Landry*, 2002; *Louchet et al.*, 2002; *Faillettaz et al.*, 2004], and components of fracture toughness [*McClung*, 2005; *Heierli et al.*, 2008].

[9] Some work claims the existence of a universal scaling exponent ($\alpha \approx 2.2 \pm 0.1$), suggesting that avalanche scaling behavior is independent of area [*Louchet*, 2001; *Faillettaz et al.*, 2006], but other studies show wide variation in

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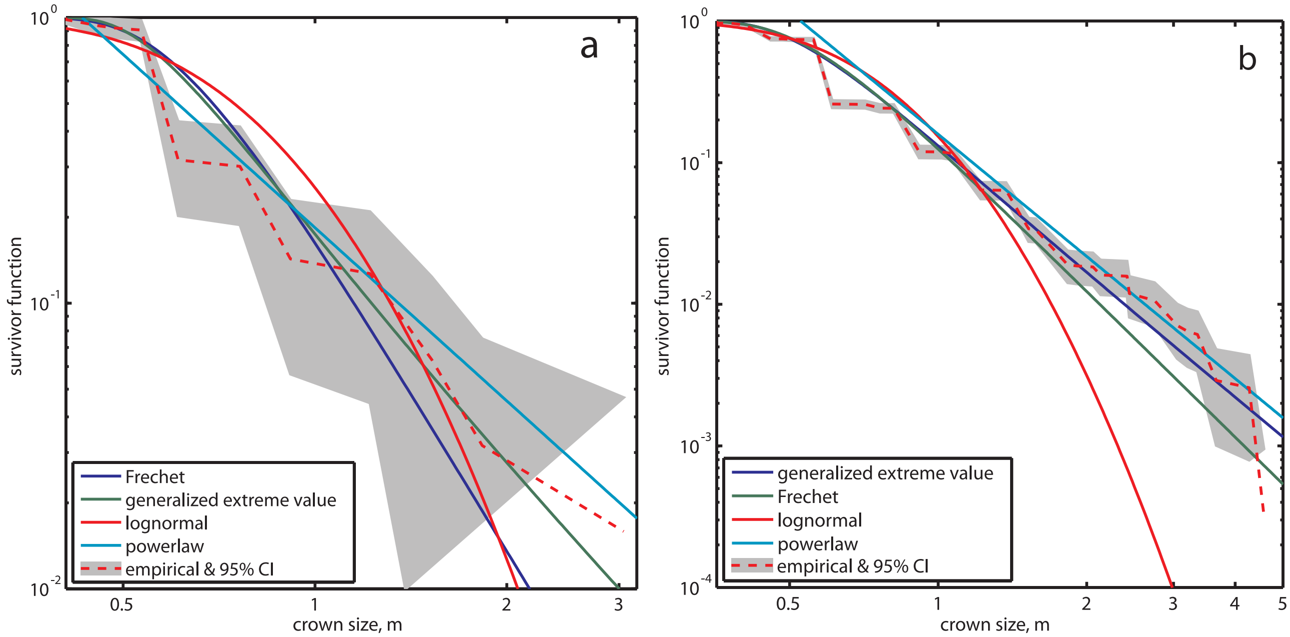

Figure 1. (a) Crowns larger than 0.305 m (N = 63, α = 3.7) for the steep Hangman's Hollow run at Mammoth Mountain. The confidence intervals (CIs) are too wide to choose a distribution. (b) All Mammoth Mountain crowns larger than 0.305 m (N = 3,106, α = 3.4). The gev and Fréchet distributions consistently stay within or near the CIs.

power-law exponents [*Birkeland and Landry*, 2002; *Rosenthal and Elder*, 2003].

[10] Open questions are: (1) At what scales does the tail index apply: path, area, region, or a universal feature? (2) What are the generating mechanisms for these distributions? (3) Are power laws in avalanches evidence of self-organized criticality [*Bak et al.*, 1988], highly optimized tolerance [*Carlson and Doyle*, 1999, 2000] or some other generating mechanism?

3. Datasets With Crown Depths

3.1. Mammoth Mountain

[11] The dataset comprises 3,106 avalanche crowns deeper than 30.5 cm (1.0 ft) on 165 avalanche paths recorded over 39 seasons at Mammoth Mountain Ski Area, CA. Based on experience of the first author (E. H. Bair) as a ski patroller there, the recorded crown depths under-represent small avalanches. Often avalanche paths are ski cut and small sloughs are triggered but not recorded. We therefore set a minimum observed depth at 30.5 cm. Moreover, bigger avalanches attract more attention and their crowns are often measured more accurately than more common shallower ones. We assume that the same distribution generating process applies to shallow and deep avalanches, and we are more interested in fitting the right-tail of the distribution than the left. We examine crown depth because it is more consistently recorded than width or length and shows larger variation than class size. Ideally, we would measure avalanche snow volume, but such a detailed measurement is operationally infeasible, so we rely on crown depth as the best proxy for volume. Patrollers also rely on crown depth, since it is the only value that can be measured quickly and accurately. On occurrence charts, sizes are recorded in inches, but even-numbered values and values corresponding to multiples or half-multiples of feet are more common in the dataset than odd numbers. Almost all Mammoth avalanches were triggered artificially.

3.2. Westwide Avalanche Network (WAN)

[12] The WAN began compiling avalanche and weather records from ski resorts and highway departments in 1968 and continued through 1995. The inventory we examined comprises 61,261 crowns deeper than 30.5 cm from 28 ski areas in continental, intermountain, and maritime ranges and one highway operation. The WAN data have similar under-reporting problems as the Mammoth data. They are recorded in feet and are reliable only for sizes greater than 30.5 cm. We smooth the WAN data to compensate for their coarseness (see Methods). Most WAN avalanches were artificially triggered.

4. Methods

[13] We fit crown depths above the threshold size discussed in Section 3. We investigate the maximum likelihood estimate (MLE) fit of seven distributions on both datasets: six with two parameters—power law, Fréchet, lognormal, Weibull, gamma, and generalized Pareto—and the generalized extreme value (gev) distribution with three parameters. Only the gev and Fréchet distributions fit all datasets, so we show these and the lognormal and power law to compare with previous work. Inverse distributions did not fit as well. The power law fits the upper tail of the distribution well, i.e. the large crowns, but not the lower tail. In the Mammoth dataset, the data are recorded for individual paths, so we explore depth distributions for paths as well as for the area. For the WAN dataset, we examine distributions of four areas with robust data collection efforts as well as the whole dataset.

[14] Since the WAN data are recorded in whole feet only, their quantization is too coarse for our fitting methods. We

Table 1. Distribution Parameters and Fits for All Mammoth Crown Sizes Above 0.305 m[a]

Distribution	α	σ (m)	μ (m)	Passed Which Test?	Probability of Fit
Generalized extreme value	2.8	0.16	0.55	rank sum	0.79
Fréchet	3.4	0.56	0	rank sum	0.71
Lognormal		0.41	−0.41	none	n/s
Power law, best KS stat	3.9		1.4	none	n/s

[a]Here n/s: not significant.

smooth the WAN crowns in the following manner: Assuming that WAN measurements in feet are rounded uniformly ±0.5 ft (i.e., 2 ft crowns are between 1.5 and 2.5 ft), we add a uniformly distributed random variable in range [0,1] and then subtract 0.5 ft. We then convert the data to meters.

[15] All but one of the distributions are fit with the MATLAB statistics toolbox [*MathWorks*, 2007], the exception being the power law, for which we use the "Santa Fe" formulas from A. Clauset et al. (Power-law distributions in empirical data, *arXiv*, 0706.1062v1, 2007) and *Newman* [2005]. Because a power law's pdf goes to infinity as x vanishes, we use a lower cutoff at 30.5 cm from the argument presented in Section 3. Crown depths do not appear to have an upper cutoff, so we do not use an upper limit [*Manning et al.*, 2005] or an exponential-decay power law [*Burroughs and Tebbens*, 2001]. Our normalized power law is a modified form of equation (2):

$$P(\geq x) = P(\geq \mu)\left(\frac{x}{\mu}\right)^{-\alpha} \quad (3)$$

[16] The survivor functions for the gev and Fréchet distributions [*MathWorks*, 2007] are:

$$\text{gev}: P(\geq x) = 1 - \exp\left\{-\left[1 + k\frac{(x-\mu)}{\sigma}\right]^{-1/k}\right\} \quad (4)$$

$$\text{Frechet}: P(\geq x) = \begin{cases} 1 & \text{for } x \leq \mu \\ 1 - \exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{-\alpha}\right] & \text{otherwise} \end{cases} \quad (5)$$

In the gev distribution, the three parameters to fit are k, σ, and μ. The Fréchet distribution is a subclass of the gev where $k > 0$. Because the survivor function for avalanches must go to 1.0 as crown size goes to zero, $\mu = 0$ in equation (5) so there are only two parameters to fit in the Fréchet distribution.

Table 2. Scaling Parameters for Four WAN Areas With the Most Robust Records of Avalanches, Compared to Mammoth Mountain

Area	Fréchet α 95% CI	gev $\alpha = 1/k$ 95% CI	Number of Crowns > 0.305 m	Which Best Fit?	Probability of Fit
Alyeska, AK	3.5–3.7	1.9–2.2	4,562	gev	0.65
Squaw Valley, CA	3.9–4.1	2.1–2.4	3,926	Fréchet	0.55
Snowbird, UT	3.6–3.8	2.4–2.9	3,704	Fréchet	0.65
Alpine Meadows, CA	4.2–4.4	2.0–2.4	3,435	gev	0.36
Mammoth Mountain, CA	3.3–3.5	2.6–3.1	3,106	gev	0.79

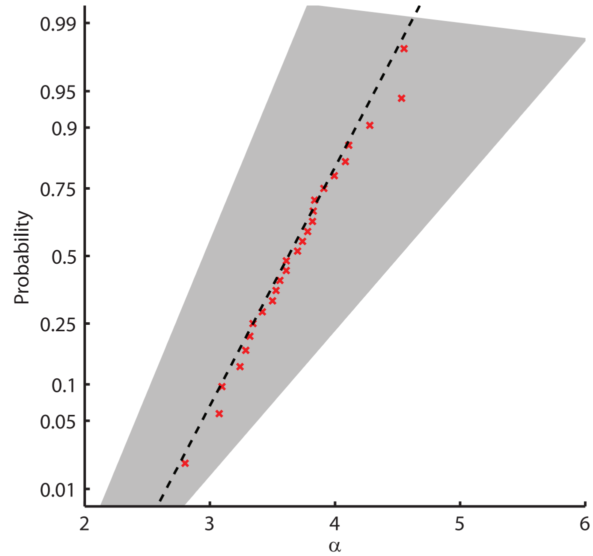

Figure 2. Normal probability plots of scaling exponents (and their upper and lower bounds) for Mammoth paths with $N \geq 40$, from the Fréchet distribution.

[17] We use three tests of statistical significance: Chi-square, Kolmogorov-Smirnov (KS), and Wilcoxon rank sum. The KS test has been shown to be effective for heavy-tailed data [*Goldstein et al.*, 2004]. Its greatest weakness for evaluating power-law fits is its sensitivity to the value of μ and its basis in the maximum deviation between the empirical and model cumulative distribution. The greatest weakness of the Chi-square test is its sensitivity to binning. The Wilcoxon rank sum test is a non-parametric alternative since it does not require assumptions about the distribution. Fits are measured using the probabilities that the null hypothesis is correct, i.e. that data are consistent with a particular distribution. Finally, 95% confidence intervals (CIs) around the empirical distribution are constructed and plotted [*Kaplan and Meier*, 1958].

5. Results

5.1. Maximum Likelihood Estimates for a Single Path

[18] We begin by examining a single avalanche path (Figure 1a). The legend is ranked with the highest average probability of a fit first. The 95% CIs of the empirical distribution are too wide to unambiguously choose a statistical distribution, but the plot suggests the gev provides the most robust fit.

5.2. MLE Estimates for All Mammoth Crowns

[19] To narrow our CIs, we examine all Mammoth crown data (Figure 1b). With the sample size dramatically increased to include all crowns, the gev and Fréchet distributions are the only ones that fit the data (Table 1) (Table 2). Table 1 also shows the tail indices α for the three scaling distributions.

5.3. Estimates of α

[20] Since maximum likelihood estimates produce asymptotically normal parameter estimates, we can compute

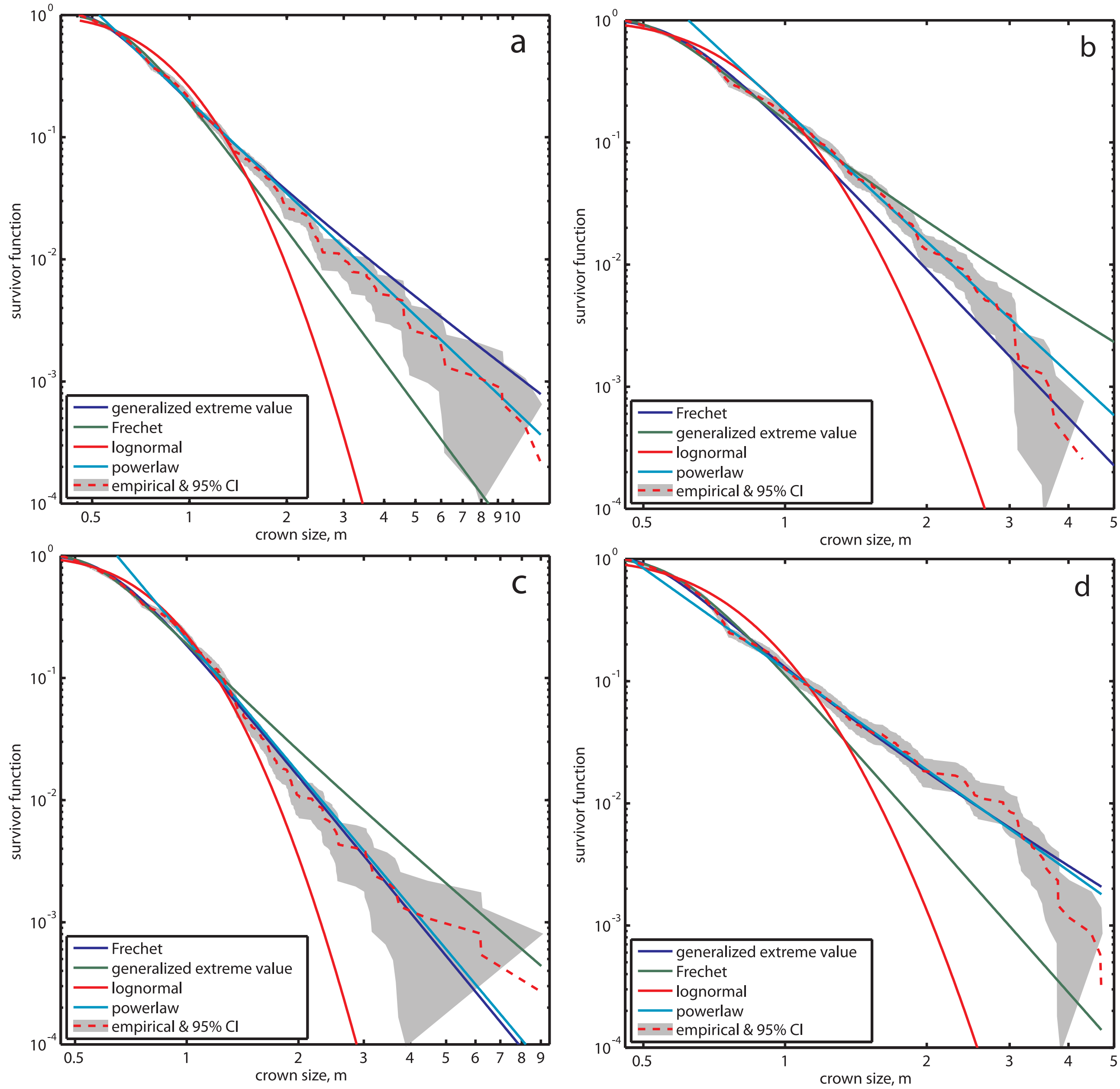

Figure 3. Statistical distributions for four WAN areas with robust data collection, for crowns deeper than 0.305 m. While the WAN data as a whole are not fit by any of the distributions, the individual areas in this figure are fit by the gev and Fréchet. This finding suggests that areas in the WAN database are a collection of maxima and their parent distributions are scaling. (a) Alyeska, AK; (b) Squaw Valley, CA; (c) Snowbird, UT; (d) Alpine Meadows, CA.

a 95% CI for α for the lumped Mammoth crowns from either the gev or Fréchet distributions. From the Fréchet, $3.3 \leq \alpha \leq 3.5$. Different paths at Mammoth show the scaling exponent α to vary from 2.8 to 4.6 among paths (Figure 2). The plots of the upper and lower bounds show significant differences among the α values even when standard error in the parameter estimates is accounted for. Since almost all avalanches examined were triggered artificially, we believe variations in α are partly caused by differences in how a path is controlled. For instance, one of the lowest α paths at Mammoth cannot be shot with artillery for safety reasons and therefore accumulates significantly more snow without avalanching than surrounding paths which can be shot with artillery. Other factors influencing α are likely: wind, exposure, slope angle, altitude, and topography.

6. Generating Mechanisms

[21] Are power laws in avalanche crown-depth distributions indicative of Self-Organized Criticality (SOC) or Highly Optimized Tolerance (HOT)? Two results argue against either conclusion. First, the exponents are too large to be consistent with either framework. In HOT models, $\alpha \sim 1/d$ where d is the dimensionality of the system. Therefore, HOT does not predict power laws with

$\alpha > 1$. Second, SOC does not predict power laws with survivor function slopes greater than 1.

[22] With the gev distribution providing the most robust fit for the Mammoth crowns, we consider a simple statistical mechanism for heavy-tailed distributions: these crown depths represent the maximum (or near maximum) size of the crown for each observed avalanche. After reviewing occurrence chart notes and crown line profiles with measured ranges of crown depth, we believe that a maximum depth, rather than the average, is typically recorded on occurrence charts and during database entry. Multiple measurements of crown height for individual avalanches are generally unavailable, so we do not know their statistical distribution. However, the gev will result from a large sample of maxima [*Gnedenko*, 1943]. Based on the shape parameter $k = 1/\alpha$ in equation (4), the gev can be broken into three maximum domains of attraction (MDAs), which for $k > 0$ is the Fréchet.

[23] Figure 3 shows smoothed crowns larger than 30.5 cm for the four largest contributors to the WAN database for deep avalanches. 95% CIs show some of the α values to be significantly different than those of Mammoth. Therefore, the generating mechanism is slightly different for different areas, and the scaling exponent is not universal as *Louchet* [2001] claims. Only the gev and Fréchet distributions significantly fit the crown heights from these four WAN areas. For the whole WAN database, the crowns are not fit by a single distribution. We suggest that the integrity of some WAN data is questionable because of varying levels of resources that areas put into their avalanche record keeping. For this reason, we limit our analysis to individual WAN areas with large numbers of deep avalanches and a robust recording protocol.

[24] The prevalence of significant fits by the gev and its special case, the Fréchet, for Mammoth and individual WAN areas suggests that these crown depths represent maxima of samples from underlying distributions. Without accurate field measurements of distributions of crown depths from individual avalanches, it is not possible to know the exact form of the generating distribution. To our knowledge, no studies to date have measured avalanche crown depths at multiple locations for a large set of avalanches.

7. Conclusion

[25] The gev and Fréchet distributions provide robust fits on path and area scales for crown depths above 30.5 cm in two large datasets. The WAN and Mammoth data comprise much larger datasets than used in previous work. The most parsimonious explanation for the observed distribution is not self-organized criticality or highly optimized tolerance, but maximum domain of attraction (MDA). In other words, there is nothing exotic or critical about these crown-depth distributions. Instead, power law behavior in the tail of the distribution results from a collection of maxima. The Fréchet MDA implies that individual avalanche crown-depth distributions follow some type of scaling distribution. More field observations on individual avalanche crown faces are needed to investigate whether individual avalanche crown depths are scaling. Given the highly variable nature of snow depth, this result would not be a surprise, taking into account the "more normal than normal" features of scaling distributions.

[26] Avalanche crown depths do not have a universal tail index. At Mammoth the tail indices for avalanche crown-depth distributions range from $\alpha \approx 2.8$ to 4.6. This finding is consistent with other geophysical phenomena, such as forest fires size distributions, which show regional variation in tail indices [*Malamud et al.*, 2005]. Future work to better understand variations in the tail index is important because paths with smaller indices have a greater proportion of large, potentially dangerous snow avalanches.

[27] **Acknowledgments.** This research was funded by NSF Grant EAR-0537327. We thank the Mammoth Mountain Ski Patrol for the avalanche data and the late Walter Rosenthal, who was instrumental in organizing them. We also thank all the U.S. Forest Service and ski area employees who collected the WAN data.

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